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ON A CERTAIN RATIONAL CUBIC TRANSFORMATION IN SPACE.*

By DR. M. W. HASKELL.

Through any point A in space can be drawn one and only one line meeting a given twisted cubic in two points. If B be the harmonic conjugate of A , the relation between the coördinates of A and B must be given by a birational transformation, and this transformation has some interesting properties.

Let the homogeneous coördinates of A be $a_0 : a_1 : a_2 : a_3$, those of B be $b_0 : b_1 : b_2 : b_3$, and let the twisted cubic be given parametrically by the formulæ

$$x_0 : x_1 : x_2 : x_3 = \lambda^3 : \lambda^2 : \lambda : 1.$$

The coördinates of any point on the line AB will be of the form

$$\mu a_0 + \nu b_0 : \mu a_1 + \nu b_1 : \mu a_2 + \nu b_2 : \mu a_3 + \nu b_3 ;$$

and, if this line is to meet the twisted cubic at all, the three equations

$$(1) \dots \left\{ \begin{array}{l} (\mu a_0 + \nu b_0)(\mu a_2 + \nu b_2) - (\mu a_1 + \nu b_1)^2 = 0 \\ (\nu a_0 + \nu b_0)(\mu a_3 + \nu b_3) - (\mu a_1 + \nu b_1)(\mu a_2 + \nu b_2) = 0 \\ (\mu a_1 + \nu b_1)(\mu a_3 + \nu b_3) - (\mu a_2 + \nu b_2)^2 = 0 \end{array} \right.$$

* Presented to the American Mathematical Society at its Chicago meeting, January 2, 1902.

must have a common root. If the line AB meet the cubic in two points, these equations must be identical and their coefficients proportional; and finally, if the points of intersection divide AB harmonically, the values of $\mu : \nu$ given by these equations must be equal and opposite and the coefficients of $\mu\nu$ must vanish:

$$(2).... \begin{cases} a_2 b_0 - 2a_1 b_1 + a_0 b_2 = 0 \\ a_3 b_0 - a_2 b_1 - a_1 b_2 + a_0 b_3 = 0 \\ a_3 b_1 - 2a_2 b_2 + a_1 b_3 = 0 \end{cases}$$

These three equations are sufficient to determine the coördinates of either point in terms of those of the other, and their symmetry shows that the transformation inverts into itself, which is also evident from the geometrical conditions. We find the following solution:

$$(3).... \begin{cases} \rho b_0 = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3 \\ \rho b_1 = a_0 a_1 a_3 - 2a_0 a_2^2 + a_1^2 a_2 \\ \rho b_2 = -a_0 a_2 a_3 + 2a_1^2 a_3 - a_1 a_2^2 \\ \rho b_3 = -a_0 a_3^2 + 3a_1 a_2 a_3 - 2a_2^3 \end{cases}$$

where ρ is an arbitrary factor indicating proportionality merely. We notice immediately that the formulae of transformation are exactly the coefficients of the cubicovariant of the binary cubic

$$a_0 x^3 + 3a_1 x^2 y + 3a_2 x y^2 + a_3 y^3,$$

while the twisted cubic curve with which we started is the common intersection of the three quadric surfaces whose equations are

$$(4).... a_0 a_2 - a_1^2 = 0, a_0 a_3 - a_1 a_2 = 0, a_1 a_3 - a_2^2 = 0,$$

whose left-hand members are the coefficients of the Hessian covariant of the binary cubic in question.

The four cubic surfaces of the transformation are tangent to each other along the given twisted cubic, and the discriminant of the binary cubic form put equal to zero:

$$(5).... (a_0 a_3 - a_1 a_2)^2 - 4(a_0 a_2 - a_1^2)(a_1 a_3 - a_2^2) = 0$$

is the equation of a quartic developable, whose cuspidal edge is the twisted cubic itself. The Jacobian of the four forms b_0, b_1, b_2, b_3 is moreover the square of this discriminant.

Since the Hessian of the cubicovariant is the product of the discriminant of the Hessian of the given binary cubic, it follows that the three quadric surfaces (4) are converted into themselves by the transformation, as indeed is every quadric surface of the form

$$\lambda(a_0a_2 - a_1^2) + \mu(a_0a_3 - a_1a_2) + \nu(a_1a_3 - a_2^2) = 0,$$

while the common intersection of all these surfaces, the given twisted cubic, is converted *point by point* into itself.

To find other special configurations under this transformation, we examine the curves of intersection (other than the twisted cubic common to them all) of the cubic surfaces $b_0=0$, $b_1=0$, $b_2=0$, $b_3=0$ taken two at a time. We find:

$b_0=0$ and $b_1=0$ are tangent to each other along the line $(a_0=0, a_1=0)$, which is moreover a double line of the surface $b_0=0$;

$b_2=0$ and $b_3=0$ are, similarly, tangent to each other along the line $(a_2=0, a_3=0)$, which is a double line of the surface $b_3=0$;

$b_0=0$ and $b_2=0$ have in common the line $(a_0=0, a_1=0)$, which of course is a double intersection, and also the line $(a_1=0, a_3=0)$;

$b_1=0$ and $b_3=0$, similarly, have in common as double intersection the line $(a_2=0, a_3=0)$ and as simple intersection the line $(a_0=0, a_2=0)$;

$b_1=0$ and $b_2=0$ have in common the three lines $(a_0=0, a_1=0)$, $(a_1=0, a_2=0)$ and $(a_2=0, a_3=0)$; and, finally,

$b_0=0$ and $b_3=0$ have in common a second twisted cubic, given parametrically by the relation

$$(6) \dots x_0 : x_1 : x_2 : x_3 = 2\lambda^3 : \lambda^2 : -\lambda : -2$$

It follows immediately (and the results can be verified without difficulty) that: the line $(a_0=0, a_1=0)$ is converted into the point $(0 : 0 : 0 : 1)$, and the line $(a_2=0, a_3=0)$ into the point $(1 : 0 : 0 : 0)$;

The lines $(a_1=0, a_3=0)$ and $(a_0=0, a_2=0)$ are interchanged; the line $(a_1=0, a_2=0)$ is converted into itself by an involution in which corresponding pairs of points are harmonically conjugate with respect to the double points $(1 : 0 : 0 : 0)$ and $(0 : 0 : 0 : 1)$;

While the line $(a_0=0, a_3=0)$ is converted into the twisted cubic (6), and vice versa.

Finally, it is geometrically evident that every point of any tangent to the twisted cubic is transformed into the point of contact of that tangent, and hence that the quartic developable (5) is transformed into the original twisted cubic, its cuspidal edge.

BERKELEY, CALIFORNIA, December 24, 1901.